Journal of Pure and Applied Algebra 215 (2011) 2315–2321

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Representation dimension and tilting

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ARTICLE INFO

Article history:
Received 7 March 2010
Received in revised form 1 November 2010
Available online 20 January 2011
Communicated by I. Reiten

MSC: 16E05; 16E10; 16G10

ABSTRACT

Let Λ be a finite dimensional k-algebra over an algebraically closed field k and let ${}_\Lambda T$ be a splitting tilting module of projective dimension at most 1. Let $\Gamma = \mathrm{End}_\Lambda T$. If the representation dimension of Λ is at most 3 then the main result asserts that the representation dimension of Γ does not exceed that of Λ.

There has been a lot of attention paid to computing the representation dimension for various classes of finite dimensional algebras Λ. The notion was introduced by Auslander in [3] in an attempt to measure the complexity of the representation theory of Λ. It seems from the results obtained in the last few years that it actually measures the homological complexity of Λ and we want to provide some further evidence.

We will not need the original definition, but rather the following characterization already going back to Auslander in [3]; see also [10] or [9] for a more detailed account.

For this let Λ be a finite dimensional k-algebra over an algebraically closed field k. Since some of the results used in Section 3 are only proved for the case where the base field is algebraically closed, we will assume this from the beginning. We will assume throughout that Λ is not semi-simple. Let $\mathrm{mod}\,\Lambda$ be the category of finitely generated left Λ-modules. Let $M \in \mathrm{mod}\,\Lambda$ be a generator–cogenerator, so ${}_\Lambda\Lambda \oplus D\Lambda_\Lambda \in \mathrm{add}\,M$, where D is the standard duality on $\mathrm{mod}\,\Lambda$ and $\mathrm{add}\,M$ is the full subcategory of $\mathrm{mod}\,\Lambda$ containing the direct sums of direct summands of M. Let d be the minimum such that there is a generator–cogenerator M with the following property: for each $X \in \mathrm{mod}\,\Lambda$ there is an exact sequence

$$0 \to M^d \to \cdots \to M^1 \to M^0 \to X \to 0 \quad \text{such that}$$
$$0 \to \mathrm{Hom}_\Lambda(M, M^d) \to \cdots \to \mathrm{Hom}_\Lambda(M, M^0) \to \mathrm{Hom}_\Lambda(M, X) \to 0$$

is exact, where $M^i \in \mathrm{add}\,M$ for $0 \le i \le d$. Then the representation dimension $\mathrm{rep.dim}\,\Lambda$ of Λ is $d + 2$. Iyama's result [19] states that the representation dimension of Λ is always finite. Trivially, if Λ is representation finite, then the representation dimension of Λ is 2. The converse was first shown by Auslander in [3] and was one of the motivations for introducing this notion. We call a generator–cogenerator where the minimum is attained an Auslander generator.

Let ${}_\Lambda T$ be a splitting tilting module (see Section 1 for a definition) with $\mathrm{proj.dim}_\Lambda T \le 1$ and let $\Gamma = \mathrm{End}_\Lambda T$. If $\mathrm{rep.dim}\,\Lambda \le 3$, then we will show, as the main result of this article, that $\mathrm{rep.dim}\,\Gamma \le \mathrm{rep.dim}\,\Lambda$. The module category of the endomorphism algebra of a splitting tilting module is 'smaller', so the main result indicates, at least in this situation, that the representation dimension is related to the complexity of the module category. In this way we underline Auslander's original intention when defining representation dimension. We point out that the representation dimension may drop, since Γ may be of finite type while Λ is of infinite type. As an application of the main result we will show that the representation dimension of a piecewise hereditary algebra is less than or equal to 3. Recall that Λ is said to be piecewise hereditary if the

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0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2010.12.012

derived category $D^b(\Lambda)$ of the category mod Λ of finitely generated left Λ-modules is equivalent as a triangulated category to $D^b(\mathcal{H})$ for some hereditary abelian category $\mathcal{H}$ (for details see Section 3). The representation dimension of piecewise hereditary algebras has also been computed in [20].

In Section 1 we will briefly review some needed aspects of tilting theory. In Section 2 we will prove the main result, while the last section contains the application to piecewise hereditary algebras.

We denote the composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in a given category $\mathcal{K}$ by fg. For unexplained representation-theoretic and derived category terminology, we refer the reader to [5,12,22].

1. A short review of tilting theory

Let Λ be a finite dimensional k-algebra over an algebraically closed field k. Recall that ${}_\Lambda T \in \operatorname{mod}\Lambda$ is called a (classical) tilting module provided the following three conditions are satisfied:

(i) $\operatorname{proj.dim}_\Lambda T \leq 1$,
(ii) $\operatorname{Ext}^1_\Lambda(T, T) = 0$ and
(iii) there is a short exact sequence $0 \to {}_\Lambda\Lambda \to T^0 \to T^1 \to 0$, with $T^0, T^1 \in \operatorname{add} T$, where $\operatorname{add} T$ is the full subcategory of mod Λ with objects the direct sums of direct summands of T.

Given a tilting module ${}_\Lambda T$ we set $\Gamma = \operatorname{End}_\Lambda T$. The Brenner–Butler theorem [16] relates mod Λ and mod Γ as follows. The bimodule ${}_\Lambda T_\Gamma$ induces torsion pairs $(\mathcal{T}(T), \mathcal{F}(T))$ on mod Λ and $(\mathcal{X}(T), \mathcal{Y}(T))$ on mod Γ, where

(i) $\mathcal{T}(T) = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(T, X) = 0\}$,
(ii) $\mathcal{F}(T) = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Hom}_\Lambda(T, X) = 0\}$,
(iii) $\mathcal{X}(T) = \{X \in \operatorname{mod}\Gamma \mid T \otimes_\Gamma X = 0\}$ and
(iv) $\mathcal{Y}(T) = \{X \in \operatorname{mod}\Gamma \mid \operatorname{Tor}^\Gamma_1(T, X) = 0\}$.

For different descriptions of these subcategories we refer the reader to [16] or [22]. In particular we will use that $\mathcal{T}(T)$ is the subcategory of mod Λ containing the Λ-modules generated by T, while $\mathcal{F}(T)$ contains the Λ-modules cogenerated by $\tau_\Lambda T$, where τ_Λ is the Auslander–Reiten translate for mod Λ.

The restriction of the functor $\operatorname{Hom}_\Lambda(T, -) : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ to $\mathcal{T}(T)$ is an equivalence from $\mathcal{T}(T) \to \mathcal{Y}(T)$ and the restriction of the functor $\operatorname{Ext}^1_\Lambda(T, -) : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ to $\mathcal{F}(T)$ is an equivalence from $\mathcal{F}(T) \to \mathcal{X}(T)$.

In the proof of the following proposition we will use the following notation. If $X, Y \in \operatorname{mod}\Lambda$, then we denote by $\underline{\operatorname{Hom}}_\Lambda(X, Y)$ the factor space of $\operatorname{Hom}_\Lambda(X, Y)$ by the subspace of those maps factoring through a projective Λ-module. We will use the following properties of a tilting module.

Proposition 1.1. *Let ${}_\Lambda T$ be a tilting module. Then*

(i) *$\tau_\Lambda T \in \mathcal{F}(T)$ and*
(ii) *$\operatorname{Ext}^1_\Lambda(\tau_\Lambda T, \tau_\Lambda T) = 0$.*

Proof. Since $\operatorname{proj.dim}_\Lambda T \leq 1$ we have by the Auslander–Reiten formula that $\operatorname{Ext}^1_\Lambda(T, T) \simeq D\operatorname{Hom}_\Lambda(T, \tau_\Lambda T)$. So we obtain the first assertion, since $\operatorname{Ext}^1_\Lambda(T, T) = 0$. Again by the Auslander–Reiten formula $\operatorname{Ext}^1_\Lambda(\tau_\Lambda T, \tau_\Lambda T) \simeq D\underline{\operatorname{Hom}}_\Lambda(\tau^-_\Lambda \tau_\Lambda T, \tau_\Lambda T)$. Now $\underline{\operatorname{Hom}}_\Lambda(\tau^-_\Lambda \tau_\Lambda T, \tau_\Lambda T)$ is an epimorphic image of $\operatorname{Hom}_\Lambda(T, \tau_\Lambda T)$, and so vanishes by the first part. □

We call a tilting module ${}_\Lambda T$ splitting if the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ on mod Γ splits, or equivalently each indecomposable Γ-module lies either in $\mathcal{X}(T)$ or in $\mathcal{Y}(T)$. We will need the following result of Hoshino [18] (for a different proof see also [12]).

Proposition 1.2. *Let ${}_\Lambda T$ be a tilting module. Then ${}_\Lambda T$ is splitting if and only if $\operatorname{inj.dim}_\Lambda X \leq 1$ for $X \in \mathcal{F}(T)$.*

Combining 1.1 and 1.2 we obtain:

Corollary 1.3. *If ${}_\Lambda T$ is a splitting tilting module, then $\tau_\Lambda T$ is a partial cotilting module.*

We will need the following description of the projective and injective Γ-modules from [16]. Let ${}_\Lambda T$ be a tilting module. Then $T = P \oplus T'$, where P is a projective Λ-module and T' does not have any indecomposable projective direct summand. Let ${}_\Lambda I = D\operatorname{Hom}_\Lambda(P, {}_\Lambda\Lambda)$.

Proposition 1.4. *Let ${}_\Lambda T$ be a tilting module with $\Gamma = \operatorname{End}_\Lambda T$. Then $\operatorname{Hom}_\Lambda(T, I) \oplus \operatorname{Ext}^1_\Lambda(T, \tau_\Lambda T')$ is a cogenerator for mod Γ and $\operatorname{Hom}_\Lambda(T, T)$ is a generator for mod Γ.*

We will use but not state explicitly the dual results and properties for cotilting modules.

2. The main result

Before showing the main result we begin with some preliminary results. But first we recall the notion of a minimal left-approximation (see [6] or [7]). Let $\mathcal{C}$ be a full subcategory of mod Λ. Let $X \in \operatorname{mod}\Lambda$; then $f : X \to F_X$ is called a left $\mathcal{C}$-approximation of X if $F_X \in \mathcal{C}$ and for all $g : X \to C$, with $C \in \mathcal{C}$, there is $h : F_X \to C$ such that $g = fh$. The map f is called a minimal left $\mathcal{C}$-approximation if F_X is of smallest length amongst all left $\mathcal{C}$-approximations of X, or equivalently φ is an automorphism whenever $f = f\varphi$ for $\varphi \in \operatorname{End}_\Lambda F_X$. The subcategory $\mathcal{C}$ is called covariantly finite in mod Λ if each $X \in \operatorname{mod}\Lambda$ admits a left $\mathcal{C}$-approximation. In this case there is also a minimal one.

The notion of a minimal right-approximation and the notion of a contravariantly finite subcategory are defined dually. If a subcategory $\mathcal{C}$ of mod Λ is both contravariantly and covariantly finite it is said to be functorially finite.

The following proposition is well-known (see for example [16,23]). For the convenience of the reader we sketch a proof.

Proposition 2.1. *Let $_\Lambda T$ be a tilting module. Then both $\mathcal{T}(T)$ and $\mathcal{F}(T)$ are functorially finite.*

Proof. Let $X \in \operatorname{mod}\Lambda$. Since $(\mathcal{T}(T), \mathcal{F}(T))$ is a torsion pair, we have a short exact sequence $0 \to t(X) \to X \to X/t(X) \to 0$, with $t(X) \in \mathcal{T}(T)$ and $X/t(X) \in \mathcal{F}(T)$. Since $\operatorname{Hom}_\Lambda(\mathcal{T}(T), \mathcal{F}(T)) = 0$, we infer that $\mathcal{T}(T)$ is contravariantly finite and $\mathcal{F}(T)$ is covariantly finite in mod Λ.

Next we consider the universal extension

$$0 \to X \to E_X \to \widetilde{T} \to 0$$

with $\widetilde{T} \in \operatorname{add} T$. This extension has the property that the connecting homomorphism $\operatorname{Hom}_\Lambda(T, \widetilde{T}) \to \operatorname{Ext}^1_\Lambda(T, X)$ is surjective. Then by construction $E_X \in \mathcal{T}(T)$ and $X \to E_X$ is a left $\mathcal{T}(T)$-approximation of X; hence $\mathcal{T}(T)$ is covariantly finite.

Finally consider the following pullback diagram of μ and g:

$$\begin{tikzcd} F_X \arrow[r, "\alpha"] \arrow[d, "f"'] & \widetilde{\tau_\Lambda T} \arrow[d, "g"'] \\ X \arrow[r, "\mu"] & I(X) \end{tikzcd},$$

where $\mu : X \to I(X)$ is the injective envelope of X and $g : \widetilde{\tau_\Lambda T} \to I(X)$ is a minimal right add $\tau_\Lambda T$-approximation of $I(X)$. Since μ is injective and the diagram is a pullback diagram we see that also α is injective. The subcategory $\mathcal{F}(T)$ is closed under submodules. Since $\tau_\Lambda T \in \mathcal{F}(T)$ we have that $F_X \in \mathcal{F}(T)$. Let $Y \in \mathcal{F}(T)$ and $\psi : Y \to X$. Since all modules in $\mathcal{F}(T)$ are cogenerated by $\tau_\Lambda T$ we have an injective map $\beta : Y \to \overline{\tau_\Lambda T}$ with $\overline{\tau_\Lambda T} \in \operatorname{add}\tau_\Lambda T$. Since $I(X)$ is Λ-injective we obtain $h : \overline{\tau_\Lambda T} \to I(X)$ with $\psi\mu = \beta h$. Since g is a minimal right add $\tau_\Lambda T$-approximation of $I(X)$ we obtain $\delta : \overline{\tau_\Lambda T} \to \widetilde{\tau_\Lambda T}$ with $h = \delta g$. Now $\beta\delta g = \beta\gamma = \psi\mu$. Thus using the pullback property we obtain a map $\varphi : Y \to F_X$ with $\psi = \varphi f$; hence $\mathcal{F}(T)$ is contravariantly finite. $\square$

The next lemma shows how to obtain a generator–cogenerator for the endomorphism algebra of a tilting module. This will be used in the proof of the main result in 2.7.

Lemma 2.2. *Let $_\Lambda T$ be a tilting module with $\Gamma = \operatorname{End}{}_\Lambda T$. Let $_\Lambda M$ be a generator–cogenerator for mod Λ. Let $\mu : M \to E_M$ be a minimal left $\mathcal{T}(T)$-approximation of M. Then $\operatorname{Hom}_\Lambda(T, T \oplus E_M) \oplus \operatorname{Ext}^1_\Lambda(T, \tau_\Lambda T)$ is a generator–cogenerator for mod Γ.*

Proof. By 2.1 we see that E_M exists. Since $D(\Lambda_\Lambda) \in \mathcal{T}(T) \cap \operatorname{add} M$ we infer that $D(\Lambda_\Lambda) \in \operatorname{add} E_M$. So the assertion follows from 1.4. $\square$

We point out that a minimal left $\mathcal{T}(T)$-approximation E of $_\Lambda\Lambda$ is contained in add T, but in general not all indecomposable direct summands of T will be direct summands of E.

The proof of the following lemma is inspired by a similar assertion in [8].

Lemma 2.3. *Let $_\Lambda T$ be a splitting tilting module with $\Gamma = \operatorname{End}{}_\Lambda T$. Let $_\Gamma S = \operatorname{Hom}_\Lambda(T, D(\Lambda_\Lambda))$. Let $_\Gamma Y \in \mathcal{Y}(T)$ and $_\Gamma X \in \mathcal{X}(T)$. Then any map $f : Y \to X$ factors over add S.*

Proof. By the Brenner–Butler theorem we know that $Y = \operatorname{Hom}_\Lambda(T, Y')$ for some $Y' \in \mathcal{T}(T)$ and $X = \operatorname{Ext}^1_\Lambda(T, X')$ for some $X' \in \mathcal{F}(T)$. Moreover $\operatorname{Hom}_\Gamma(Y, X) \simeq \operatorname{Ext}^1_\Lambda(Y', X')$. Let $0 \to Y' \to I(Y') \to Z' \to 0$ be exact with $I(Y')$ injective. Since by Proposition 1.2 $\operatorname{inj.dim}_\Lambda X' \le 1$ we infer that $\operatorname{Ext}^1_\Lambda(I(Y'), X') \to \operatorname{Ext}^1_\Lambda(Y', X')$ is surjective. Thus also $\operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(T, I(Y')), X) \to \operatorname{Hom}_\Gamma(Y, X)$ is surjective, so f factors over add S. $\square$

In the proof of the main result in this section we will also need the following three statements. The proof of the first uses an idea from [16].

Lemma 2.4. *Let $_\Lambda T$ be a splitting tilting module. A minimal right add $\tau_\Lambda T$-approximation $F_X \to X$ for $X \in \mathcal{F}(T)$ is injective.*

Proof. Let $X \in \mathcal{F}(T)$ and let $f : F_X \to X$ be a minimal right add $\tau_\Lambda T$-approximation. Assume that f is not injective. Then $F_X \to \operatorname{im} f$ is a proper surjection. Also $\operatorname{im} f \in \mathcal{F}(T)$, since $\mathcal{F}(T)$ is closed under images. Now $\operatorname{im} f$ is cogenerated by $\tau_\Lambda T$, so there is an injective map $\mu : \operatorname{im} f \to \widetilde{\tau_\Lambda T}$ with $\widetilde{\tau_\Lambda T} \in \operatorname{add}\tau_\Lambda T$. Also $\ker f \in \mathcal{F}(T)$, for $\mathcal{F}(T)$ is closed under submodules. Now $\operatorname{Ext}^2(\operatorname{cok}\mu, \ker f) = 0$, since $\operatorname{inj.dim}_\Lambda \ker f \le 1$. Thus $\operatorname{Ext}^1_\Lambda(\operatorname{cok}\mu, F_X) \to \operatorname{Ext}^1_\Lambda(\operatorname{cok}\mu, \operatorname{im} f)$ is surjective. Thus we obtain the following commutative diagram of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & F_X \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \operatorname{cok}\mu \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \operatorname{im} f \arrow[r] & \widetilde{\tau_\Lambda T} \arrow[r] & \operatorname{cok}\mu \arrow[r] & 0 \end{tikzcd}$$

In particular this yields a short exact sequence

$$0 \to F_X \to \operatorname{im} f \oplus E \to \widetilde{\tau_\Lambda T} \to 0.$$

By 1.1 we know that $\operatorname{Ext}^1_\Lambda(\tau_\Lambda T, \tau_\Lambda T) = 0$. So this sequence splits and so $\operatorname{im} f \in \operatorname{add} \tau_\Lambda T$. It is easy to see that also $\operatorname{im} f \to X$ is a right add $\tau_\Lambda T$-approximation of X, contradicting the minimality of f; hence f is injective. $\square$

The next result is well-known, but we will give a proof for the convenience of the reader.

Lemma 2.5. *Let ${}_\Lambda T$ be a tilting module and let $X \in \operatorname{mod} \Lambda$. If $\beta : X \to F$ is a left $\mathcal{T}(T)$-approximation of X, then β is injective. If $\mu : X \to E_X$ is a minimal left $\mathcal{T}(T)$-approximation of X, then $Q = \operatorname{cok} \mu \in \operatorname{add} T$. Moreover, if $\alpha : X \to E$ is a left $\mathcal{T}(T)$-approximation of X with $\overline{Q} = \operatorname{cok} \alpha \in \operatorname{add} T$, then $E \simeq E_X \oplus T'$ for some $T' \in \operatorname{add} T$.*

Proof. Since $D(\Lambda_\Lambda) \in \mathcal{T}(T)$ we clearly have that β is injective. So we have an exact sequence

$$0 \to X \to E_X \to Q \to 0$$

Since $E_X \in \mathcal{T}(T)$ and $\mathcal{T}(T)$ is closed under factor modules, we see that $Q \in \mathcal{T}(T)$. If μ is minimal, then Wakamatsu's lemma (see [4]) says that $\operatorname{Ext}^1_\Lambda(Q, \mathcal{T}(T)) = 0$. For this we use that $\mathcal{T}(T)$ is closed under extensions. Since $Q \in \mathcal{T}(T)$ there is a short exact sequence

$$0 \to K \to \widetilde{T} \to Q \to 0,$$

with $\widetilde{T} \in \operatorname{add} T$ and $K \in \mathcal{T}(T)$. Thus the sequence splits and therefore $Q \in \operatorname{add} T$.

Let $\alpha : X \to E$ be a left $\mathcal{T}(T)$-approximation of X with $\overline{Q} = \operatorname{cok} \alpha \in \operatorname{add} T$. So by the first part of the proof we have that α is injective. Since $E \in \mathcal{T}(T)$ there is $\psi : E_X \to E$ with $\alpha = \mu\psi$. Thus we obtain the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\mu"] \arrow[d, equal] & E_X \arrow[r] \arrow[d, "\psi"'] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r, "\alpha"] & E \arrow[r, "\beta"] \arrow[d, "\pi"'] & \overline{Q} \arrow[r] \arrow[d, "\pi'"'] & 0 \\
 & & Z \arrow[r, equal] & Z &
\end{tikzcd}$$

Since α is a left $\mathcal{T}(T)$-approximation and $E_X \in \mathcal{T}(T)$ there is also $\varphi : E \to E_X$ with $\mu = \alpha\varphi$. So $\mu = \alpha\varphi = \mu\psi\varphi$. By the minimality of μ we see that $\psi\varphi$ is an automorphism. Thus ψ is a split mono, and so π is a split epi. Hence there is ε with $\varepsilon\pi = \mathrm{id}_Z$. But then $\varepsilon\beta\pi' = \varepsilon\pi = \mathrm{id}_Z$, so π' is a split epi. Thus $\overline{Q} \simeq Z \oplus Q$; in particular $Z \in \operatorname{add} T$. Hence $E \simeq E_X \oplus T'$ for some $T' \in \operatorname{add} T$. $\square$

We will use the following consequence which shows that approximations behave well with respect to direct sums. Again we will give the argument.

Corollary 2.6. *Let ${}_\Lambda T$ be a tilting module and let $X \in \operatorname{mod} \Lambda$. Moreover let $\mu : X \to E_X$ be a minimal left $\mathcal{T}(T)$-approximation of X. Let $Y \in \operatorname{add} X$ and $\mu' : Y \to E$ be a left $\mathcal{T}(T)$-approximation of Y with $\operatorname{cok} \mu' \in \operatorname{add} T$; then $E \in \operatorname{add}(E_X \oplus T)$.*

Proof. In fact, let $X = \oplus_{i=1}^r X_i$ with X_i indecomposable. Consider a minimal left $\mathcal{T}(T)$-approximation of $\mu_i : X_i \to E_{X_i}$ of X_i. Then by 2.5 we have that $\operatorname{cok} \mu_i \in \operatorname{add} T$. Now $F = \oplus_{i=1}^r E_{X_i}$ clearly is a left $\mathcal{T}(T)$-approximation of X, so again by 2.5 we conclude that $F \in \operatorname{add}(E_X \oplus T)$. Let E_Y be a minimal left $\mathcal{T}(T)$-approximation of Y. Then by 2.5 $E \simeq E_Y \oplus T'$ for some $T' \in \operatorname{add} T$. Let $Y = \oplus_{i=1}^s X_i$ with X_i indecomposable. Let $G = \oplus_{i=1}^s E_{X_i}$. Then clearly G is a left $\mathcal{T}(T)$-approximation of Y, and thus $G \simeq E_Y \oplus T''$ for some $T'' \in \operatorname{add} T$. But $\operatorname{add} G \subset \operatorname{add} F$ shows that $E_Y \in \operatorname{add}(E_X \oplus T)$ and hence $E \in \operatorname{add}(E_X \oplus T)$. $\square$

We will now show the main result of this section.

Theorem 2.7. *Let Λ be a finite dimensional algebra of representation dimension at most 3 and let ${}_\Lambda T$ be a splitting tilting module with $\Gamma = \operatorname{End} {}_\Lambda T$. Then rep.dim $\Gamma \leq$ rep.dim Λ.*

Proof. If rep.dim $\Lambda = 2$, then Λ is representation finite. Since ${}_\Lambda T$ is splitting, we clearly have that Γ is representation finite and so rep.dim $\Gamma = 2$. So we may assume that Λ is representation infinite. If Γ is representation finite the assertion holds, so we may also assume that Γ is representation infinite.

Let ${}_\Lambda M$ be an Auslander generator for mod Λ. Let $\mu_M : M \to E_M$ be a minimal left $\mathcal{T}(T)$-approximation of M. By 2.5 we have that $\operatorname{cok} \mu_M \in \operatorname{add} T$.

We will show that ${}_\Gamma N = \operatorname{Hom}_\Lambda(T, T \oplus E_M) \oplus \operatorname{Ext}^1_\Lambda(T, \tau_\Lambda T)$ is an Auslander generator for mod Γ, so for $X \in \operatorname{mod} \Gamma$ there is a short exact sequence

$$0 \longrightarrow N^1 \longrightarrow N^0 \xrightarrow{\pi} X \longrightarrow 0$$

with $N^0, N^1 \in \operatorname{add} N$ and π a right $\operatorname{add} N$-approximation of X. By 2.2 we know that ${}_{\Gamma}\Gamma \oplus D(\Gamma_{\Gamma}) \in \operatorname{add} N$, so ${}_{\Gamma}N$ is a generator–cogenerator for $\operatorname{mod} \Gamma$.

Clearly we may assume that X is indecomposable. Since ${}_{\Lambda}T$ is splitting it is enough to distinguish the following two cases.

First assume that $X \in \mathcal{X}(T)$. So we know that $X = \operatorname{Ext}^1_{\Lambda}(T, X')$ for some $X' \in \mathcal{F}(T)$. We consider the universal extension

$$0 \to X' \to E \to D(\Lambda_{\Lambda})^r \to 0$$

where $r = \dim_k \operatorname{Ext}^1_{\Lambda}(D(\Lambda_{\Lambda}), X')$. It is straightforward to check that E is injective. In fact, $\operatorname{inj.dim}_{\Lambda} E \leq 1$, since $\operatorname{inj.dim}_{\Lambda} X' \leq 1$. And by construction we have that $\operatorname{Ext}^1_{\Lambda}(D(\Lambda_{\Lambda}), E) = 0$. Let $\mu : \widetilde{\tau_{\Lambda} T} \to X'$ be a minimal right $\operatorname{add} \tau_{\Lambda} T$-approximation of X'. By 2.4 we know that μ is injective. Thus we obtain the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\tau_{\Lambda} T} \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r] & E \arrow[r] & D(\Lambda_{\Lambda})^r \arrow[r] & 0
\end{tikzcd}$$

Note that E' is injective, for $\operatorname{inj.dim}_{\Lambda} \tau_{\Lambda} T \leq 1$. As abbreviations we set $S = \operatorname{Hom}_{\Lambda}(T, D(\Lambda_{\Lambda}))$ and $\overline{S} = \operatorname{Hom}_{\Lambda}(T, E)$. By applying the tilting functors we obtain the following commutative diagram of short exact sequences of Γ-modules:

$$\begin{tikzcd}
0 \arrow[r] & \overline{S} \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\Lambda}(T, E') \arrow[r] \arrow[d] & \operatorname{Ext}^1_{\Lambda}(T, \widetilde{\tau_{\Lambda} T}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{S} \arrow[r] & S^r \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

Thus we obtain a short exact sequence of Γ-modules:

$$0 \longrightarrow \operatorname{Hom}_{\Lambda}(T, E') \longrightarrow S^r \oplus \operatorname{Ext}^1_{\Lambda}(T, \widetilde{\tau_{\Lambda} T}) \xrightarrow{\pi} X \longrightarrow 0.$$

Note that $\operatorname{Hom}_{\Lambda}(T, E') \oplus S^r \oplus \operatorname{Ext}^1_{\Lambda}(T, \widetilde{\tau_{\Lambda} T}) \in \operatorname{add} N$.

Since $\operatorname{Ext}^1_{\Lambda}(D(\Lambda_{\Lambda}), X) \simeq \operatorname{Hom}_{\Gamma}(S, X)$ and $\mathcal{F}(T) \simeq \mathcal{X}(T)$ we infer that π is a right $\operatorname{add} S \oplus \operatorname{Ext}^1_{\Lambda}(T, \tau_{\Lambda} T)$-approximation. Now 2.3 implies that π is a right $\operatorname{add} N$-approximation.

Next we assume that $X \in \mathcal{Y}(T)$. So we know that $X = \operatorname{Hom}_{\Lambda}(T, X')$ for some $X' \in \mathcal{T}(T)$. Since M is an Auslander generator for $\operatorname{mod} \Lambda$ we have by assumption a short exact sequence

$$0 \longrightarrow M^1 \longrightarrow M^0 \xrightarrow{\pi} X' \longrightarrow 0$$

with $M^0, M^1 \in \operatorname{add} M$ and π a right $\operatorname{add} M$-approximation of X'. We consider the following commutative diagram of short exact sequences of Λ-modules:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M^1 \arrow[r] \arrow[d, "\mu_{M^1}"'] & M^0 \arrow[r, "\pi"] \arrow[d, "\mu"'] & X' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E_{M^1} \arrow[r, "\alpha"] \arrow[d, "\varepsilon"'] & E \arrow[r, "\beta"] \arrow[d, "\varepsilon'"'] & X' \arrow[r] & 0 \\
 & Z \arrow[r, equal] \arrow[d] & Z \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where μ_{M^1} is a minimal left $\mathcal{T}(T)$-approximation of M^1. By 2.6 and 2.5 we see that $E_{M^1} \in \operatorname{add}(E_M \oplus T)$ and $Z \in \operatorname{add} T$; hence μ is a left $\mathcal{T}(T)$-approximation of M^0. So by 2.6 we have that $E \in \operatorname{add}(E_M \oplus T)$. We claim that β is an $\operatorname{add}(E_M \oplus T)$-approximation of X'. We have that $\operatorname{Hom}_{\Lambda}(T, E) \to \operatorname{Hom}_{\Lambda}(T, X')$ is surjective, since $E_{M^1} \in \mathcal{T}(T)$.

Applying $\operatorname{Hom}_{\Lambda}(M, -)$ to the diagram above yields the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Hom}_{\Lambda}(M, M^0) \arrow[r, "\operatorname{Hom}_{\Lambda}(M,\pi)"] \arrow[d] & \operatorname{Hom}_{\Lambda}(M, X') \arrow[d, equal] \\
\operatorname{Hom}_{\Lambda}(M, E) \arrow[r, "\operatorname{Hom}_{\Lambda}(M,\beta)"] & \operatorname{Hom}_{\Lambda}(M, X')
\end{tikzcd}$$

By assumption we have that $\operatorname{Hom}_{\Lambda}(M, \pi)$ is surjective and so we see that $\operatorname{Hom}_{\Lambda}(M, \beta)$ is surjective.

By construction we have an exact sequence

$$0 \longrightarrow M \xrightarrow{\mu_M} E_M \xrightarrow{\gamma} Q \longrightarrow 0$$

with $Q \in \operatorname{add} T$.

Applying $\operatorname{Hom}_\Lambda(-, E)$ and $\operatorname{Hom}_\Lambda(-, X')$ yields the following commutative diagram of exact sequences where we abbreviate $\operatorname{Hom}_\Lambda(U, V)$ as (U, V) and $\operatorname{Hom}_\Lambda(f, V)$ as (f, V). Note that $\operatorname{Ext}^1_\Lambda(Q, E \oplus X') = 0$, since $E \oplus X' \in \mathcal{T}(T)$ and $Q \in \operatorname{add} T$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (Q, E) & \xrightarrow{(\gamma,E)} & (E_M, E) & \xrightarrow{(\mu_M,E)} & (M, E) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle (Q,\beta)} & & \Big\downarrow{\scriptstyle (E_M,\beta)} & & \Big\downarrow{\scriptstyle (M,\beta)} & & \\ 0 & \longrightarrow & (Q, X') & \xrightarrow{(\gamma,X')} & (E_M, X') & \xrightarrow{(\mu_M,X')} & (M, X') & \longrightarrow & 0 \end{array}$$

Now (Q, β) is surjective, since $\operatorname{Ext}^1_\Lambda(Q, E_{M^1}) = 0$. We have seen above that (M, β) is surjective; hence (E_M, β) is surjective. Thus β is an add $(E_M \oplus T)$-approximation of X.

Applying $\operatorname{Hom}_\Lambda(T, -)$ to the short exact sequence

$$0 \longrightarrow E_{M^1} \xrightarrow{\alpha} E \xrightarrow{\beta} X' \longrightarrow 0$$

yields a short exact sequence of Γ-modules

$$0 \longrightarrow (T, E_{M^1}) \xrightarrow{(T,\alpha)} (T, E) \xrightarrow{(T,\beta)} X \longrightarrow 0$$

Clearly (T, E_{M^1}) and (T, E) belong to $\operatorname{add} N$. Since $(\mathcal{X}(T), \mathcal{Y}(T))$ is a torsion pair we see that $\operatorname{Hom}_\Gamma(\mathcal{X}(T), X) = 0$, for $X \in \mathcal{Y}(T)$. We know that $\mathcal{T}(T) \simeq \mathcal{Y}(T)$. So we conclude that (T, β) is a left add N-approximation of X. This finishes the proof of the theorem. $\square$

We will use in the next section also the dual of 2.7 for splitting cotilting modules.

3. Piecewise hereditary algebras

Piecewise hereditary algebras are a well studied class of algebras; see for example [1,12,15,14,17]. We refer the reader to earlier work on the representation dimension for some classes of piecewise hereditary algebras; see [2,8,21].

In the introduction we have defined piecewise hereditary algebras. The hereditary abelian category $\mathcal{H}$ occurring in the definition is called the type of Λ. Note that the type is only defined up to derived equivalence. It follows from [13] that there are up to derived equivalence two classes, namely $\mathcal{H} = \operatorname{mod} H$ for a finite dimensional hereditary k-algebra H and $\mathcal{H} = \operatorname{coh} \mathbb{X}$ for a weighted projective line $\mathbb{X}$ in the sense of [11]. We will need the following results which are contained in [15,14].

Theorem 3.1. *Let Λ be a piecewise hereditary k-algebra.*

(i) *If the type of Λ is $\operatorname{mod} H$ for a finite dimensional hereditary k-algebra H, then there exists a sequence of finite dimensional algebras Λ_i and splitting tilting modules ${}_{\Lambda_i}T_i$, for $0 \le i \le m$, such that $H = \Lambda_0$, $\Lambda_{i+1} = \operatorname{End}_{\Lambda_i} T_i$ and $\Lambda = \Lambda_m$.*
(ii) *If the type of Λ is $\operatorname{coh} \mathbb{X}$ for a weighted projective line $\mathbb{X}$, then there is a quasitilted algebra Γ and a sequence of finite dimensional algebras Λ_i and splitting tilting or cotilting modules ${}_{\Lambda_i}T_i$, for $0 \le i \le m$, such that $\Gamma = \Lambda_0$, $\Lambda_{i+1} = \operatorname{End}_{\Lambda_i} T_i$ and $\Lambda = \Lambda_m$.*

As an application of this and the main result of the previous section we now show a bound for the representation dimension of a piecewise hereditary algebra. Note that this yields a different proof for the result shown in [8] for piecewise hereditary algebras of type H for a finite dimensional hereditary k-algebra H, or equivalently iterated tilted algebras.

Corollary 3.2. *Let Λ be a piecewise hereditary k-algebra. Then $\operatorname{rep.dim} \Lambda \le 3$.*

Proof. If Λ is of type $\operatorname{mod} H$ for a finite dimensional hereditary k-algebra H, the result follows from 3.1(i) and 2.7 by using the well-known fact that $\operatorname{rep.dim} H \le 3$ for a finite dimensional hereditary algebra H; see [3].

If Λ is of type $\operatorname{coh} \mathbb{X}$ for a weighted projective line $\mathbb{X}$, the result follows from 3.1(ii), 2.7 and its dual on using that $\operatorname{rep.dim} \Gamma \le 3$ for a quasitilted algebra [21]. $\square$

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